Solution 7

1. Consider maps from $\mathbb{R}$ to itself. Provide explicit examples of continuous maps with exactly one, two and three fixed, and one map satisfying $|f(x)-f(y)| < |x-y|$ but no fixed points.

   Solution. Let f be our function. We consider $g(x) = f(x) - x$. It suffices to produce examples with exactly one, two and three roots. For instance, $g_1(x) = -x$ has exactly one root. $g_2(x) = x^2 - 1$ has exactly two roots. $g_3(x) = (x-1)(x-2)(x-3)$ has exactly three roots. The corresponding f_1, f_2, f_3 fulfil our requirement. Finally, the function $f(x) = x + \log(1 + e^{-x})$ does not have any fixed point.

2. Let T be a continuous map on the complete metric space X. Suppose that for some k, T^k becomes a contraction. Show that T admits a unique fixed point. This generalizes the contraction mapping principle in the case $k = 1$.

   Solution. Since T^k is a contraction, there is a unique fixed point $x \in X$ such that $T^k x = x$. Then $T^{k+1}x = T^k T x = Tx$ shows that Tx is also a fixed point of T^k. From the uniqueness of fixed point we conclude $Tx = x$, that is, x is a fixed point for T. Uniqueness is clear since any fixed point of T is also a fixed point of T^k.

3. Show that the equation $2x \sin x - x^4 + x = 0.001$ has a root near $x = 0$.

   Solution. Here $\Psi(x) = 2x \sin x - x^4$. We need to find some r, γ so it is a contraction. We have
$$\Psi'(x) = 2 \sin x + 2x \cos x - 4x^2.$$
   Using $|\sin x| \le |x|$ and $|\cos x| \le 1$, we have
$$|\Psi'(x)| \le 2r + 2r + 4r^2 = 4r(1 + r^2) \ , \quad \forall x, |x| \le r \ .$$
   By the mean-value theorem, $\Psi(x_1) - \Psi(x_2) = \Psi'(z)(x_1 - x - 2)$ where z lies on the line segment from x_1 to x_2 (so $|z| \le r$ when $|x_1|, |x_2| \le r$). Therefore,
$$|\Psi(x_1) - \Psi(x_2)| = |\Psi(z)||x_1 - x_2| \le 2r(1 + r^2)|x_1 - x_2| \ .$$
   Now we fix $r = 1/5$. Then $\gamma = 4r(1 + r^2) = 0.812$ and $R = (1 - \gamma)r = 0.0336$. By the theorem on the perturbation of identity, the equation $x + \Psi(x) = a$ is solvable whenever $|a| \le 0.0336$. Now, $0.001 < 0.0336$, so the equation has a root x satisfying $|x| \le 1/5 = 0.2$.

4. Study the solvability of
$$\sin^2 \pi x + 2x^2 = 2.0012 \ .$$
   Hint: Consider $\Phi(1) = 2$ and shift things to the origin as done in Notes.

   Solution. The equation can be expressed as $\Phi(x) = 2.0012$. Observe that $\Phi(1) = 2$. We let $\Phi_1(x) = \Phi(x + 1) - \Phi(1) = \sin^2 \pi x + 2x^2 + 4x$. Now, $4x$ is not the form of identity, but in any case we have reduced the problem to
$$\sin^2 \pi x + 2x^2 + 4x = 0.0012.$$
   To solve this new equation, which is the same as
$$\frac{1}{4} \sin^2 \pi x + \frac{1}{2} x^2 + x = 0.0003,$$

is in the form of perturbation of identity. Let $\Psi(x) = \frac{1}{4}\sin^2 \pi x + \frac{1}{2}x^2$ be the perturbation term. For $x, |x| \le r$, we have $\Psi'(x) = \frac{\pi}{4}\sin 2\pi x + x$, so $|\Psi'(x)| \le \pi^2 r/2 + r$ after using $\sin 2\pi x| \le 2\pi|x|$. Therefore, by the mean value theorem, there is some z lying between x_1 and x_2 (so $|z| \le r$ too)

$$|\Psi(x_1) - \Psi(x_2)| = |\Psi'(z)||x_1 - x_2| \le \frac{\pi^2 r}{2} + r .$$

If now we choose $r = 1/(4\pi^2)$, then $\gamma = \pi^2 r/2 + r = 1/2 + 1/(4\pi^2) < 1$. By the theorem on perturbation of identity, the equation $x + \Psi(x) = a$ is solvable for $a, |a| \le R$, where

$$R = \left(\frac{1}{2} - \frac{1}{4\pi^2}\right)\frac{1}{4\pi^2} .$$

Now, we check that $0.0003 < R$, so $x + \Psi(x) = 0.0012$ is solvable, which in turn implies that the original problem is also solvable.

5. Can you solve the system of equations

$$x + y^4 = 0, \quad y - x^2 = 0.015 \ ?$$

Solution. Here we work on $\mathbb{R}^2$ and $\Phi(x,y) = (x,y) + \Psi(x,y)$ where $\Psi(x,y) = (\Psi_1(x,y), \Psi_2(x,y)) = (-y^4, x^2)$. We have

$$\frac{\partial \Psi_1}{\partial x} = 0, \quad \frac{\partial \Psi_1}{\partial y} = -4y^3 , \quad \frac{\partial \Psi_2}{\partial x} = 2x , \quad \frac{\partial \Psi_2}{\partial y} = 0 .$$

It follows that

$$|\Psi(p) - \Psi(q)| \le \sqrt{16y^6 + 4x^2}|p - q| \le 2r\sqrt{1 + 4r^2}|p - q| ,$$

where $p = (x_1, y_1), q = (x_2, y_2)$, and $|p - q|$ is the Euclidean distance between p and q. We choose $r = 1/3$, so $\gamma = 2r\sqrt{1 + 4r^2} = 0.6996$. Then $R = (1 - \gamma)r = 0.2$. Now, the point $(0, 0.015)$ satisfies $|(0, 0.015)| = 0.015 < 0.2$, so this system has a solution (x, y) satisfying $|(x, y)| < 1/3$.

Note. Here we have used the discussion on page 8 in the revised notes: Ψ is a contraction when

$$\sqrt{\sum_{i,j} (\partial \Psi_i / \partial x_j)^2} < 1 .$$

6. Can you solve the system of equations

$$x + y - x^2 = 0, \quad x - y + xy \sin y = -0.005 \ ?$$

Solution. First we rewrite the system in the form of $I + \Psi$. Indeed, by adding up and subtracting the equations, we see that the system is equivalent to

$$x + (-x^2 + xy \sin y)/2 = -0.0025, \quad y + (-x^2 - xy \sin y)/2 = 0.0025 .$$

Now we can take

$$\Psi(x, y) = \frac{1}{2}(-x^2 + xy \sin y, -x^2 - xy \sin y) ,$$

and proceed as in the previous problem.

7. Show that the integral equation

$$y(x) = \alpha e^x - \int_0^1 \frac{\sin x}{3-t}\, y^3(t)\, dt\ ,$$

is solvable for sufficiently small α. Give an estimate on the smallness of α.

Solution. We take

$$\Phi(y)(x) = y(x) + \int_0^1 \frac{\sin x}{3-t} y^3(t)\, dt\ ,$$

so that

$$\Psi(y)(x) = \int_0^1 \frac{\sin x}{3-t} y^3(t)\, dt\ .$$

We have

$$\begin{aligned} |\Psi(y_1)(x) - \Psi(y_2)(x)| &= \left| \int_0^1 \frac{\sin x}{3-t} (y_1^3(t) - y_2^3(t))\, dt \right| \\ &\le \int_0^2 \frac{1}{3-t} (y_1^2(t) + |y_1(t)y_2(t)| + y_2^2(t)) |y_1(t) - y_2(t)|\, dt \\ &\le \frac{1}{2} \int_0^1 (y_1^2(t) + |y_1(t)y_2(t)| + y_2^2(t))\, dt \|y_1 - y_2|_\infty\ , \end{aligned}$$

after noting $1/(3-t) \le 1/2$ for $t \in [0,1]$. Therefore, for $y \in B_r(0)$,

$$\|\Psi(y_1) - \Psi(y_2)\|_\infty \le \frac{3r^2}{2} \|y_1 - y_2\|_\infty\ ,$$

and Ψ is a contraction as long as $3r^2/2$ is less than 1. Let us choose $r = \sqrt{2}/2$ so that $3r^2/2 = 3/4 < 1$. By using the theorem on Perturbation of Identity, the equation $\Phi(y) = \alpha e^x$ is solvable in $B_r(0)$ as long as $\|\alpha e^x\|_\infty \le R = (1-3/4)r = \sqrt{2}/8$. As $e^x \le e$ on $[0,1]$, we conclude that whenever $|\alpha| \le \sqrt{2}e^{-1}/8$, this integral equation is solvable.

8. Let $A = (a_{ij})$ be an $n \times n$ matrix. Show that the matrix $I + A$ is invertible if $\sum_{i,j} a_{ij}^2 < 1$. Give an example showing that $I + A$ could become singular when $\sum_{i,j} a_{ij}^2 = 1$.

Solution. Let $\Phi(x) = Ix + Ax$ so $\Psi(x) = Ax$ for $x \in \mathbb{R}^n$. We have

$$\|\Psi(x_1) - \Psi(x_2)\|_2 = \|A(x_1 - x_2)\|_2 \le \|A\| \|x\|_2\ .$$

As explained in Notes, we have the following estimate on the operator norm, $\|A\| \le \sum_{i,j} a_{ij}^2$. Take $\gamma = \sum_{i,j} |a_{ij}|$. If $\sum_{i,j} a_{ij} < 1$, Ψ is a contraction and there is only one root of the equation $\Phi(x) = 0$ in the ball $B_r(0)$. However, since we already know $\Phi(0) = 0$, 0 is the unique root. Now, we claim that $I + A$ is non-singular, for there is some $z \in \mathbb{R}^n$ satisfying $(I + A)z = 0$, we can find a small number α such that $\alpha z \in B_r(0)$. By what we have just shown, $\alpha z = 0$ so $z = 0$, that is, $I + A$ is non-singular and thus invertible.

Note. See how linearity plays its role in this problem.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be C^2 and $f(x_0) = 0, f'(x_0) \neq 0$. Show that there exists some $\rho > 0$ such that

$$Tx = x - \frac{f(x)}{f'(x)}, \quad x \in (x_0 - \rho, x_0 + \rho),$$

is a contraction. This provides a justification for Newton's method in finding roots for an equation.

Solution. $T'(x) = \dfrac{f(x)f''(x)}{f'(x)^2}$. Since f is C^2 and $f(x_0) = 0, f'(x_0) \neq 0$, it follows that T is C^1 in a neighborhood of x_0 with $T(x_0) = x_0, T'(x_0) = 0$ and there exists $\rho > 0$

$$|T'(x)| < 1, \quad x \in (x_0 - 2\rho, x_0 + 2\rho),$$

As a result, T is a contraction in $[x_0 - \rho, x_0 + \rho]$.

10. Consider the iteration

$$x_{n+1} = \alpha x_n(1 - x_n), \ x_1 \in [0, 1] \ .$$

Find

(a) The range of α so that $\{x_n\}$ remains in $[0, 1]$.

(b) The range of α so that the iteration has a unique fixed point 0 in $[0, 1]$.

(c) Show that for $\alpha \in [0, 1]$ the fixed point 0 is attracting in the sense: $x_n \to 0$ whenever $x_0 \in [0, 1]$.

Solution. Let $Tx = \alpha x(1-x)$. The max of T attains at $1/2$ so the maximal value is $\alpha/4$. Therefore, the range of α is $[0, 4]$ so that T maps $[0, 1]$ to itself. Next, 0 is always a fixed point of T. To get no other, we set $x = \alpha x(1-x)$ and solve for x and get $x = (\alpha-1)/\alpha$. So there is no other fixed point if $\alpha \in [0, 1]$. Finally, it is clear that T becomes a contraction when $\alpha \in [0, 1)$, so the sequence $\{x_n\}$ with $x_0 \in [0, 1]$, $x_n = T^n x_0$, always tends to 0 as $n \to \infty$. Although T is not a contraction when $\alpha = 1$, one can still use elementary mean (that is, $\{x_n\}$ is always decreasing,) to show that 0 is an attracting fixed point.

11. Show that every continuous function from $[0, 1]$ to itself admits a fixed point. Here we don't need it a contraction. Suggestion: Consider the sign of $g(x) = f(x) - x$ at $0, 1$ where f is the given function.

Solution. Let $f \in C[0, 1]$. Clearly, if $f(0) = 0$, then 0 is a fixed point. So assume $f(0) \neq 0$. Here we take $f(0) > 0$. Consider the continuous function $g(x) = f(x) - x$. We have $g(0) = f(0) > 0$ and $g(1) = f(1) - 1 \leq 0$. If equality holds, then $f(1) = 1$, 1 is a fixed point. If inequality holds, that is, $g(1) < 0$, by the mean-value theorem there is some $\xi \in (0, 1)$ such that $g(\xi) = 0$, that is, $f(\xi) - \xi = 0$, so ξ is a fixed point. The case $f(0) < 0$ can be handled similarly.

Note. This example shows that every continuous function from $[0, 1]$ to itself, not only contractions, admits a least one fixed point. (But not necessarily unique.) Similar result holds for all continuous maps on a compact, convex subset in $\mathbb{R}^n$ to itself. It is called Brouwer's fixed point theorem.